# Minimum Weight Design of Truss Structures with Geometric Nonlinear Behavior

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**The paper presents an optimization method based on optimality criterion for minimum weight design of structures with geometric nonlinear behavior. The nonlinear critical load is determined by finding the load level at which the Hessian of the potential energy ceases to be positive definite. A recurrence relation based on the criterion that at optimum the nonlinear strain energy density be equal in all the members is used to develop an algorithm. Sample problems are given to illustrate the application of the method to truss type structures with a large number of design variables.**

## Introduction

A NUMBER of algorithms based on the optimality criterion approach have been developed to design a minimum weight structure with specified constraints on nodal displacements, element stresses, system stability, etc.[1,2] The optimization algorithm consisted of analyzing the structure by the finite element method and using a recurrence relation derived from the appropriate optimality criterion to modify the design variables. The optimality criterion was derived by differentiating the Lagrangian with respect to the design variables. The displacements were assumed to be small and, in the finite element analysis, linear equilibrium equations were solved to determine the response of the structure to the applied loads. In the case of system stability the constraints were defined (see Refs. 3-6) by the associated linear eigenvalue problem. This definition of system stability for some structures may not be valid because of the nonlinear behavior of the structure which may be due to the geometry of the structure or the presence of geometric imperfections. The correct procedure for these structures would be to analyze the structure by using nonlinear equilibrium equations. This will be particularly true for large space structures (LSS). In this paper an optimization method based on the optimality criterion approach is presented for structures with geometric nonlinear behavior.

In the case of a structure optimized with constraints on linear stability the optimum structure can have more than one critical buckling mode. This design tends to become imperfection sensitive and a small deviation in the geometry of the structure can reduce its load carrying capacity substantially. The imperfection sensitivity of the optimized structure can be reduced by designing the structure so that buckling loads associated with the critical buckling modes are not equal (see Ref. 7). A real structure has geometric imperfections and these structures tend to have nonlinear behavior. The degree of nonlinearity depends on the geometry and the applied load. If the postbuckling curve is stable, then the effect of imperfection is not drastic in reducing the load carrying capacity of the structure. In the case of three-dimensional truss structures it has been shown in Ref. 7 that the reduction in the load carrying capacity from the linear buckling load is primarily due to the nonlinear behavior and secondarily due to the system geometric imperfections. It is more appropriate to optimize a structure on the basis of a nonlinear stability constraint when it has an inherent tendency to have nonlinear behavior. In Ref. 8 the problem of maximization of critical load (limit point stability) of shallow space trusses with constant weight was investigated. The optimization was carried out by using nonlinear mathematical programming. The method proposed in this reference was not suitable for structures with a large number of design variables because of difficulties in evaluating the gradient of the constraint on the system limit point. The present method overcomes this difficulty by the use of recurrence relations based on the nonlinear strain energy density distribution in the structure.

In the subsequent sections a short description of the nonlinear analysis method used in the algorithm is given. This is followed by the derivation of the optimality criterion and the recurrence relations. The application of the method is then illustrated by optimizing a number of truss problems.

## Nonlinear Analysis

Geometric nonlinear analysis of structures results from the inclusion of nonlinear terms in the strain-displacement relations. This has the same implication as that of writing the equations of equilibrium on the deformed geometry of the structure.

For a structure built up from three-dimensional truss elements, a nonlinear strain displacement relation is obtained by employing the following definition for strain

$$\epsilon = (\ell - \ell_0)/\ell_0 \tag{1}$$

where $\ell_0$ and $\ell$ are the undeformed and deformed lengths of the truss element shown in Fig. 1. Although the two ends $p$ and $q$ of the truss element of Fig. 1 are allowed to undergo a large and finite displacement, the strain resulting from such a displacement is assumed to be small enough to permit a linear stress-strain relation. This implies that the strain energy of a

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typical $i$th truss element can be expressed as

$$e_i = \tfrac{1}{2} E_i A_i \ell_i \epsilon_i^2 \tag{2}$$

where $E_i$ is the Young's modulus of the material, $A_i$ the cross-sectional area, and $\ell_i$ the length of the $i$th element. In Eq. (2) $\epsilon_i$ is related to the displacements of the two ends by

$$\epsilon_i = \left[ 1 + \frac{1}{\ell_{0i}^2} \sum_{j=1}^{3} 2(\Delta x_j)(\Delta u_j) + (\Delta u_j)(\Delta u_j) \right]^{1/2} \tag{3}$$

where $\Delta$ is the difference operator for the $q$ and $p$ end values of the coordinates $x_1$, $x_2$, $x_3$ and the global displacements $u_1$, $u_2$, and $u_3$.

The total potential energy of a structure built up for $n$ such elements with a total of $m$ degrees of freedom can now be conveniently expressed as

$$\pi = \sum_{i=1}^{n} e_i - \{u\}^t\{P\} = U - \{u\}^t\{P\} \tag{4}$$

where $\{u\}$ is a vector of the global displacements of the nodes of the assembled structure and $\{P\}$ the vector of the externally applied forces at these nodes.

Equations of equilibrium of the structure are provided by the principle of stationary value of the total potential energy and are obtained by setting the first variation of $\pi$ with respect to the global displacement $\{u\}$ to zero. Thus,

$$\frac{\partial \pi}{\partial\{u\}} = \frac{\partial U}{\partial\{u\}} - \{P\} = \{0\} \tag{5}$$

Equations (5) represent a highly nonlinear system of equations. For purposes of stability analysis, it is convenient to rewrite such equations in the form

$$\frac{\partial U}{\partial\{u\}} - \beta\{p\} = \{0\} \tag{6}$$

where $\beta$ is a load parameter and $\{p\}$ represents a specified distribution of loads that must be scaled to produce instability.

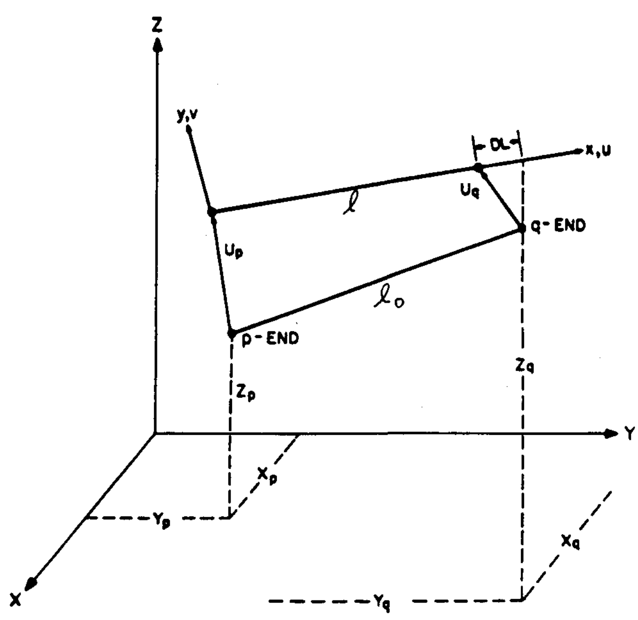


**Fig. 1 Geometry and deformation of the truss element.**

Although several direct methods for solving such equations are available[9] a convenient method is the direct minimization of $\pi$ using techniques of mathematical programming to determine all the stable equilibrium configurations. Of interest to the present investigation, however, was an accurate location of a critical point of the structure. A critical point, which may be either a limit or a bifurcation point, is characterized by the value of the load parameter $\beta$ which causes a loss of positive definiteness of the Hessian matrix, $[\partial^2\pi/\partial u_i \partial u_j]$ evaluated at the equilibrium configuration. That is to say, it is that load level at which

$$\det \left| \frac{\partial^2 \pi}{\partial u_i \partial u_j} \right| = 0 \tag{7}$$

The location of such a point is accomplished very efficiently by using a globally convergent quasi-Newton method described in Ref. 10.

## Optimality Criterion

Nonlinear critical load is defined as the load level at which the condition in Eq. (7) is satisfied. The form of this equation is not suitable to define a constraint relation. The constraint relation should be such that it can be explicitly differentiable with respect to the design variables in order to derive the explicit optimality criterion. This makes it necessary to use other equivalent conditions which are also satisfied at the nonlinear critical point. In addition to the condition in Eq. (7) at the critical point, 1) the total potential energy is stationary and 2) the equilibrium equations are satisfied. These conditions will be used to derive the optimality criterion. In the optimization problem considered here, the geometry of the structure is given, the load distribution applied to the structure is specified, and the design variables are the cross-sectional areas of the members. The structure is idealized with truss elements and carry only axial load.

The optimization problem can be defined as:

Minimize the weight

$$W = \sum_{i=1}^{n} \rho_i A_i \ell_i \tag{8}$$

subject to

$$g_1 = \pi - \bar{\pi} = 0 \tag{9}$$

where $\pi$ is the total energy and $\bar{\pi}$ is the total potential energy associated with the optimum design at the nonlinear critical point. This implicitly assumes that there are no discontinuous changes in the critical deformation patterns with changes in area. In Eq. (8) $\rho_i$ is the density of the material, $\ell_i$ the length of the element, and $n$ the number of elements.

Using Eqs. (8) and (9) the Lagrangian can be written as

$$L = \sum_{i=1}^{n} \rho_i A_i \ell_i - \lambda_1 (\pi - \bar{\pi}) \tag{10}$$

where $\lambda_1$ is the Lagrange multiplier. The optimality criterion can be obtained by differentiating Eq. (10) with respect to the design variable $A_i$ and setting the resulting equations to zero. This gives

$$\rho_i \ell_i - \lambda_1 \left( \frac{\partial \pi}{\partial A_i} + \sum_{j=1}^{m} \frac{\partial \pi}{\partial u_j} \frac{\partial u_j}{\partial A_i} \right) i = 1, \ldots, n \tag{11}$$

where $m$ is the number of displacement components. By virtue of Eq. (5), corresponding to every displacement degree $u_j$,

$\partial\pi/\partial u_j$ vanishes and, hence, Eq. (11) can be written as

$$\rho_i \ell_i - \lambda_I \frac{\partial \pi}{\partial A_i} = 0 \tag{12}$$

In the total potential energy $\pi$, the nonlinear strain energy associated with the $i$th element is a function of the design variable $A_i$ only. Therefore, using Eq. (2) the gradient of the total potential energy can be written as

$$\frac{\partial \pi}{\partial A_i} = \frac{e_i}{A_i} \tag{13}$$

Substitution of this in Eq. (12) gives

$$\rho_i \ell_i - \lambda_I \frac{e_i}{A_i} = 0 \tag{14}$$

or

$$1 = \lambda_I \frac{e_i}{\rho_i \ell_i A_i} \tag{15}$$

or

$$1 = \lambda_I \frac{\tilde{e}_i}{\rho_\theta} \tag{16}$$

In Eq. (16), $\tilde{e}_i$ is the nonlinear strain energy density. Equation (16) states that in an optimum structure the ratio of nonlinear strain energy density to mass density is equal for all elements. This is a rather heuristic derivation of the optimality criterion. However, on the basis of the results presented in Ref. 8, where such a criterion has been more rigorously established for simple truss structure, it seems only natural that the effectiveness of such a criterion for a minimum weight design of more complex shallow space truss structures be examined. This has been the objective of the present investigation.

## Recurrence Relation

The optimality criterion derived in the last section provided information on the distribution of the nonlinear strain energy density in an optimum design. In order to obtain a design satisfying the criterion, an iterative scheme can be used. This consists of changing the design variables by using the recurrence relation after determining the nonlinear critical point and associated deformed shape of the structure by nonlinear analysis. The recurrence relation can be written by using the optimality criterion. By multiplying both sides of Eq. (16) by $(A_i)^r$ and taking its $r$th root, one obtains

$$A_i^{\nu+1} = A_i^{\nu} \left( \lambda_I \frac{\tilde{e}_i}{\rho_i} \right)_{\nu}^{1/r} \tag{17}$$

where $\nu+1$ and $\nu$ are the iteration numbers. The step size parameter is $r$ and can be changed by assigning proper values.

### Scaling of the Design

The specified load on the structure must be equal to the nonlinear critical load of the structure. If this condition is satisfied after each iteration, then every design will be a feasible one. This can be achieved by scaling the design after the analysis phase in the iteration cycle. The relative design variable $\tilde{A}_i$ can be written as

$$A_i = \Lambda \tilde{A}_i \tag{18}$$

Strain energy is a linear function of the design variables $A_i$, even in the geometric nonlinear case. Hence, scaling the areas of all the members of the structure by a factor $\Lambda$ has the effect of scaling the load parameter $\beta$ by the same scale factor $\Lambda$ with no change in the displacement pattern. This is obvious by examining Eq. (6) wherein the terms of $\partial U/\partial\{u\}$, although nonlinear functions of the displacements, are just linear in member areas. Hence,

$$\Lambda = \frac{\beta_{\text{specified}}}{\beta_{\text{critical}}} \tag{19}$$

## Optimization Procedure

A computer program was written by using portions of the ACTION[10] program as an analysis module and recurrence relation and the scaling equation derived in the last two sections. The major steps of the optimization procedure are as follows.

1) Assign initial values to the design variables.

2) Select a set of load levels for nonlinear analysis.

3) Analyze the structure at different load levels and find the upper bound for the nonlinear critical load.

4) Determine the nonlinear critical load within the specified tolerance, say $\delta\beta$, within which the Hessian matrix of the potential energy ceases to be positive definite.

5) Determine the scaling parameter $\Lambda$ by using Eq. (19) so that this nonlinear critical load is equal to the specified load level $\{P\}$. Scale all the design variables by using Eq. (18).

6) Determine the weight of the structure. If the weight is less than that of the previous iteration, then go to the next step; otherwise, reduce the step size and go to step 3 with member areas of the previous minimum weight design.

7) Using the displacement vector $\{u\}$ associated with the critical load evaluate $\tilde{e}_i$ for each element.

8) Modify the design variables by using the recurrence relation with the assumption that the Lagrange multiplier is equal to unity. This assumption is permissible since the design variables are normalized at each iteration with the maximum value of a design variable being equal to unity.

9) Go to step 3, or quit, if the convergence criterion is satisfied.

## Illustrative Examples

In all the problems discussed in this section, Young's modulus ($E$) of 68.94 GPa ($10^7$ psi) and material density ($\rho$) of 2768 kg/cm$^3$ (0.1 lb/in.$^3$) were assumed. The minimum size for all elements was set at 0.645 cm$^2$ (0.1 in.$^2$). The minimum size constraint was treated as a passive constraint. If the recurrence relation reduced the size of any element to a value smaller than the minimum size, then the cross-sectional area of that element was increased to the minimum size. In the interest of investigating the effectiveness of the algorithm in obtaining a minimum weight design of a structure with nonlinear load-displacement behavior, no other constraints were imposed. The step size parameter $r$ in Eq. (17) was set equal to 4 and, whenever the weight of the structure increased, the parameter was doubled to reduce the step size. Except for the two-bar truss problems, the initial areas of all the elements for all other problems were assumed to be equal. The structure was scaled at each iteration so that the nonlinear critical load was equal to the design load.

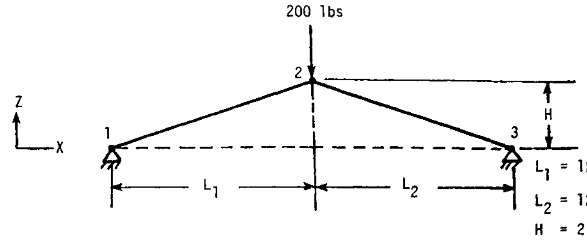


Fig. 2 Two-bar truss (symmetric).

Table 1 Iteration history for two-bar truss (symmetric)

| Iteration No. | Weight, kg | (lb) | Iteration No. | Weight, kg | (lb) |
|---|---|---|---|---|---|
| 1 | 82.91 | (182.7874) | 6 | 73.56 | (162.4819) |
| 2 | 75.93 | (167.4052) | 7 | 73.70 | (162.4783) |
| 3 | 74.25 | (163.7000) | 8 | 73.70 | (162.4774) |
| 4 | 73.73 | (162.5534) | 9 | 73.69 | (162.4773) |
| 5 | 73.70 | (162.4963) | 10 | 73.69 | (162.4771) |

Table 2 Iteration history for two-bar truss (unsymmetric)

| Iteration No. | Weight, kg | (lb) | Iteration No. | Weight, kg | (lb) |
|---|---|---|---|---|---|
| 1 | 32.61 | (71.9532) | 6 | 30.22 | (66.6247) |
| 2 | 30.80 | (67.9125) | 7 | 30.22 | (66.6211) |
| 3 | 30.36 | (66.9399) | 8 | 30.21 | (66.6201) |
| 4 | 30.25 | (66.6993) | 9 | 30.21 | (66.6199) |
| 5 | 30.25 | (66.6396) | 10 | 30.21 | (66.6198) |

Table 3 Iteration history for four-bar truss

| Iteration No. | Weight, kg | (lb) | Iteration No. | Weight, kg | (lb) |
|---|---|---|---|---|---|
| 1 | 54.73 | (120.6535) | 6 | 52.18 | (115.0421) |
| 2 | 52.80 | (116.4181) | 7 | 52.18 | (115.0378) |
| 3 | 52.33 | (115.3804) | 8 | 52.18 | (115.0374) |
| 4 | 52.21 | (115.1225) | 9 | 52.17 | (115.0369) |
| 5 | 52.19 | (115.0582) | 10 | 52.17 | (115.0368) |

Table 4 Relative strain energy density distribution for the four-bar shallow truss

| Element | Initial design | Final design |
|---|---|---|
| 1 | 0.5582 | 0.9982 |
| 2 | 0.3285 | 0.9968 |
| 3 | 0.3565 | 0.9969 |
| 4 | 1.0 | 1.0 |

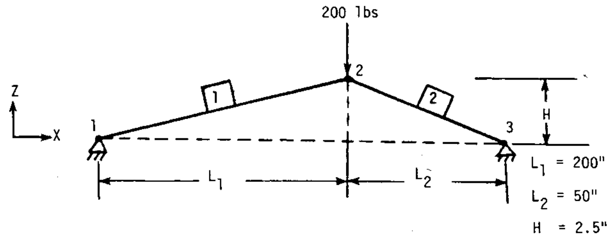


Fig. 3 Two-bar truss (unsymmetric).

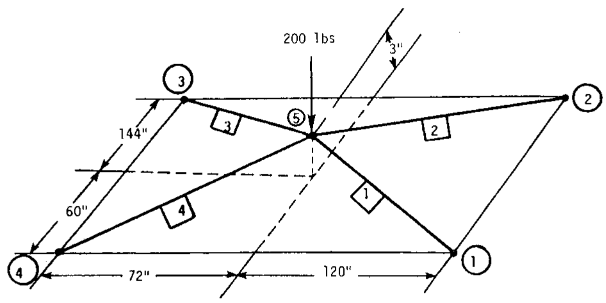


Fig. 4 Four-bar shallow truss.

**Example 1 Two-Bar Truss (Symmetric)**

As a simple application of the proposed algorithm a symmetric two-bar truss shown in Fig. 2 was considered. The structure was to be designed for a load of 90.72 kg (200 lb) applied at node 2 in the negative $z$ direction. Because of symmetry, this structure must have equal cross-sectional areas for both members at the optimum. Therefore, initial relative cross-sectional areas of the two members were assumed to be 2 and 1. For a design load of 90.72 kg (200 lb), the initial weight of the structure was 82.91 kg (182.7874 lb) with the areas of the two members equal to 62.88 cm$^2$ (9.7467 in.$^2$) and 31.44 cm$^2$ (4.8733 in.$^2$). The iteration history for the structure is given in Table 1. After ten iterations the weight was reduced to 73.70 kg (162.4711 lb) and the cross-sectional areas of the two members were both equal to 41.92 cm$^2$ (6.4977 in.$^2$). The exact solution to this problem under the assumption of $H/L_i \ll 1$ is given in Ref. 8 where the cross-sectional areas of the members were found to be 41.90 cm$^2$ (6.49519 in.$^2$). It should be noted that the finite element solution does not make any assumption regarding the orders of $H/L_i$, $i=1,2$.

**Example 2 Two-Bar Truss (Unsymmetric)**

The unsymmetric two-bar truss shown in Fig. 3 was optimized with initial relative areas of members 1 and 2 being equal to 2 and 1, respectively. The design load was again 90.72 kg (200 lb). For this design load, the initial design weighed 36.63 kg (71.9532 lb) with cross-sectional areas of members 1 and 2 equal to 20.62 and 10.31 cm$^2$ (3.1973 and 1.5986 in.$^2$), respectively. The iteration history is given in Table 2. The optimum design weighed 30.21 kg (66.6198 lb) with cross-sectional areas of members 1 and 2 equal to 17.187 and 17.182 cm$^2$ (2.6641 and 2.6633 in.$^2$), respectively. In Ref. 8 it is shown that for this unsymmetric truss the optimum design weighed 30.17 kg (66.5277 lb) and the cross-sectional areas of both the members were equal to 17.164 cm$^2$ (2.6605 in.$^2$). It is interesting to note that for an unsymmetric structure the cross-sectional areas of both elements are equal at the optimum. Comparing the results of the symmetric and unsymmetric two-bar truss problems it is even more interesting to note that the weight of the symmetric structure is greater than that of the unsymmetric structure for the same design load.

**Example 3 Four-Bar Shallow Truss**

The four-bar truss in Fig. 4 was optimized with the initial areas of all members equal for a design load of 90.72 kg (200 lb). For this design load, the initial weight of the structure was 54.72 kg (120.6535 lb) with the cross-sectional area of each member equal to 13.504 cm$^2$ (2.0929 in.$^2$). The iteration history is given in Table 3. The optimum design weighed 52.17 kg (115.0368 lb) and the cross-sectional areas of members 1-4 were found to be 13.91, 10.75, 11.09, and 13.536 cm$^2$ (2.1555, 1.6668, 1.7202, and 2.0894 in.$^2$), respectively. The distribution of the nonlinear strain energy densities for the initial and final designs is given in Table 4. The solution obtained here is just one of the many optimum designs which can be obtained for this structure (see Ref. 8).

**Example 4 Shallow Truss**

The truss shown in Fig. 5 has 46 members and is subjected to vertical design loads of 136.08, 544.31, and 136.07 kg (300, 1200, and 300 lb) at nodes 7, 13, and 19, respectively. The coordinates of the node points are given in Table 5. The iteration history for this structure is given in Table 6. The step-size parameter $r$ used in each iteration is also given in the table. The step-size parameter was increased in order to reduce the step size whenever the weight of the structure increased above that of the previous iteration. For the

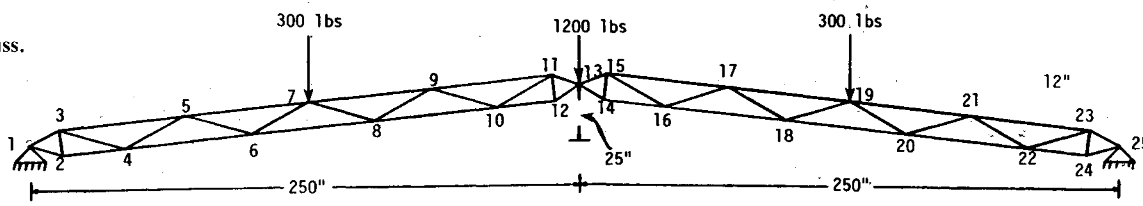


Fig. 5 Shallow truss.

**Table 5 Coordinates of the node points shallow truss, in. (m)**

| Node[a] | X | Y |
|---|---|---|
| 1 | −10.94 (−0.27) | 4.94 (0.13) |
| 2 | 0.0 (0.0) | 0.0 (0.0) |
| 3 | −1.19 (−0.03) | 11.94 (0.30) |
| 4 | 28.66 (0.73) | 2.87 (0.07) |
| 5 | 56.14 (1.42) | 17.67 (0.44) |
| 6 | 85.99 (2.18) | 8.59 (0.22) |
| 7 | 113.50 (2.88) | 23.41 (0.59) |
| 8 | 143.30 (3.63) | 14.33 (0.36) |
| 9 | 170.80 (4.33) | 29.14 (0.74) |
| 10 | 200.70 (5.09) | 20.07 (0.51) |
| 11 | 228.10 (5.79) | 34.87 (0.88) |
| 12 | 229.30 (5.82) | 22.93 (0.58) |
| 13 | 239.10 (6.07) | 29.94 (0.76) |

[a] Coordinates of other node points can be found by symmetry.

**Table 6 Iteration history for shallow truss**

| Iteration No. | Weight, kg (lb) | Parameter $r$ |
|---|---|---|
| 1 | 153.62 (338.6844) | 4 |
| 2 | 117.16 (258.3000) | 4 |
| 3 | 107.61 (237.2446) | 4 |
| 4 | 107.16 (236.2522) | 8 |
| 5 | 106.61 (235.0274) | 8 |
| 6 | 106.45 (234.6862) | 8 |
| 7 | 106.28 (234.3078) | 8 |
| 8 | 106.27 (234.3032) | 8 |
| 9 | 106.25 (234.2450) | 8 |
| 10 | 106.23 (234.2066) | 8 |
| 15 | 106.20 (234.1344) | 8 |
| 17 | 106.19 (234.1310) | 16 |

**Table 7 Areas of the members for the minimum weight design of shallow truss**

| Element | Connecting nodes | Area[a], $cm^2$ | ($in.^2$) |
|---|---|---|---|
| 1 | 1-3 | 15.965 | (2.4747) |
| 2 | 1-2 | 15.287 | (2.3695) |
| 3 | 2-3 | 7.636 | (1.1837) |
| 4 | 3-4 | 1.683 | (0.2609) |
| 5 | 3-5 | 15.286 | (2.3694) |
| 6 | 2-4 | 13.258 | (2.0551) |
| 7 | 4-5 | 1.457 | (0.2259) |
| 8 | 5-6 | 1.436 | (0.2227) |
| 9 | 5-7 | 17.745 | (2.7512) |
| 10 | 4-6 | 10.550 | (1.6353) |
| 11 | 6-7 | 1.254 | (0.1945) |
| 12 | 7-8 | 1.252 | (0.1941) |
| 13 | 7-9 | 17.718 | (2.7463) |
| 14 | 6-8 | 8.226 | (1.2751) |
| 15 | 8-9 | 1.438 | (0.2230) |
| 16 | 9-10 | 1.450 | (0.2249) |
| 17 | 9-11 | 15.257 | (2.3649) |
| 18 | 8-10 | 10.549 | (1.6351) |
| 19 | 10-11 | 1.671 | (0.2590) |
| 20 | 10-12 | 13.241 | (0.0524) |
| 21 | 11-12 | 7.584 | (1.1756) |
| 22 | 12-13 | 15.242 | (2.3626) |
| 23 | 11-13 | 15.927 | (2.4687) |

[a] The areas of the members on other half of the structure can be found by symmetry.

**Table 8 Relative strain energy density distribution for the shallow truss**

| Element | Initial design | Final design |
|---|---|---|
| 1 | 0.7248 | 0.9987 |
| 2 | 0.5926 | 0.9933 |
| 3 | 0.1459 | 0.9579 |
| 4 | 0.1329 | 0.9034 |
| 5 | 0.7095 | 0.9874 |
| 6 | 0.4429 | 0.9866 |
| 7 | 0.0094 | 0.9093 |
| 8 | 0.0055 | 0.9368 |
| 9 | 1.0 | 1.0 |
| 10 | 0.2210 | 0.9749 |
| 11 | 0.0037 | 0.9273 |
| 12 | 0.0037 | 0.9269 |
| 13 | 0.9962 | 0.9999 |
| 14 | 0.1196 | 0.9514 |
| 15 | 0.0055 | 0.9368 |
| 16 | 0.0093 | 0.9873 |
| 17 | 0.7066 | 0.9873 |
| 18 | 0.2212 | 0.9749 |
| 19 | 0.0131 | 0.9033 |
| 20 | 0.0055 | 0.9865 |
| 21 | 0.1439 | 0.9576 |
| 22 | 0.5889 | 0.9932 |
| 23 | 0.7214 | 0.9886 |

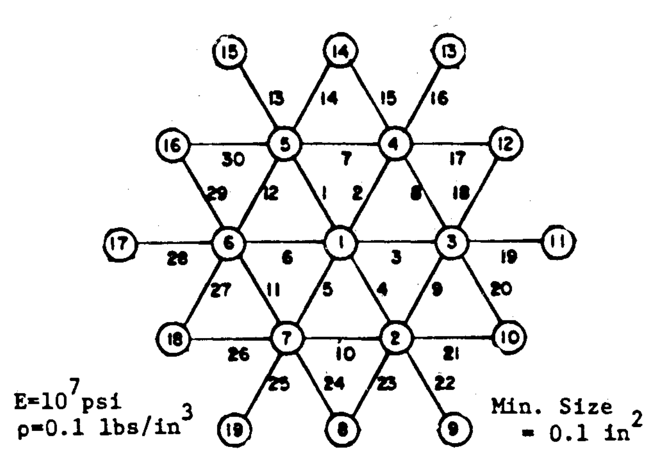


Fig. 6 Plan view of dome structure.

specified design loads, the initial weight of the structure with the cross-sectional areas of all elements equal to 13.95 $cm^2$ (2.1638 $in.^2$) was 153.62 kg (338.69 lb). The weight of the structure after three iterations was 107.61 kg (237.2446 lb) and it was not substantially reduced with additional iterations. The cross-sectional areas of the members for the optimum design weighing 106.25 kg (234.1310 lb) and the element connections are given in Table 7. Table 8 contains the strain energy density distributions for the initial and optimum designs.

#### Example 5 Dome Structure

The 30 member three-dimensional dome structure shown in Fig. 6 was optimized for a concentrated load of 907.2 kg (2000 lb) applied in the vertically downward direction at node

**Table 9 Coordinates of the node points of dome structure, in. (m)**

| Node | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| 1 | 0.0 | 0.0 | 85.912 (2.18) |
| 3 | 360.0 ( 9.14) | 0.0 | 64.662 (1.64) |
| 4 | 180.0 ( 4.57) | 311.769 ( 7.92) | 64.662 (1.64) |
| 11 | 720.0 (18.29) | 0.0 | 0.0 |
| 12 | 540.0 (13.72) | 311.769 ( 7.92) | 21.709 (0.55) |
| 13 | 360.0 ( 9.14) | 623.538 (15.83) | 0.0 |
| 14 | 0.0 | 623.538 (15.83) | 21.709 (0.55) |

**Table 10 Iteration history for dome structure**

| Iteration No. | Weight kg | (lb) |
|---|---|---|
| 1 | 753.32 | (1660.7960) |
| 2 | 445.95 | (983.1602) |
| 3 | 396.98 | (875.2096) |
| 4 | 381.13 | (840.2418) |
| 5 | 371.44 | (818.9048) |
| 6 | 365.09 | (804.9048) |
| 7 | 360.69 | (795.1926) |
| 8 | 357.55 | (788.2850) |
| 9 | 355.29 | (783.2366) |
| 10 | 353.56 | (779.4680) |
| 11 | 352.26 | (776.6074) |
| 15 | 349.40 | (770.3064) |
| 20 | 348.11 | (767.4552) |
| 25 | 347.70 | (766.5488) |
| 30 | 347.53 | (766.1880) |

**Table 11 Relative strain energy density distribution for the dome structure**

| Elements | Initial design | Final design |
|---|---|---|
| 1,2,3,4, 5,6 | 1.000 | 0.9998 |
| 7,8,9,10 11,12 | 0.6506 | 1.0000 |
| 13,16,19, 22,25,28 | 0.0383 | 0.9979 |
| 14,15,17, 18,20,21 23,24,26 27,29,30 | 0.0000 | 0.2634 |

1 and with a minimum size constraint of 0.645 cm$^2$ (0.1 in.$^2$). The coordinates of the node points are given in Table 9. For the specified design load, the initial weight with areas of all members equal to 9.85 cm$^2$ (1.5280 in.$^2$) was 753.32 kg (1660.7960 lb). The iteration history for this dome structure is given in Table 10. The weight of the structure after 10 iterations was 353.56 kg (779.4680 lb). With an additional 20 iterations the weight of the structure was reduced to 347.54 kg (766.1880 lb). The step size parameter $r$ was equal to 4 for all the iterations. The strain energy density distribution for the initial and the optimum design with a weight of 347.54 kg (766.1880 lb) is given in Table 11. The cross-sectional areas of the members for the optimum design are given in Table 12.

## Conclusions

It has been demonstrated that the algorithm based on the optimality criterion of uniform nonlinear strain energy density distribution can be used to design a minimum weight structure with nonlinear load-displacement behavior. In the illustrative problems the design load and the nonlinear critical load were equal. However, the structure can be designed for

**Table 12 Areas of the members for the minimum weight design dome structure**

| Element | Area cm$^2$ | (in.$^2$) |
|---|---|---|
| 1,2,3,4, 5,6, | 10.92 | (1.6926) |
| 7,8,9,10, 11,12 | 8.87 | (1.3754) |
| 13,16,19 22,25,28 | 1.74 | (0.2693) |
| 14,15,17, 18,20,21,23,24,26, 27,29,30 | 0.645 | (0.1000) |

some percentage of the nonlinear critical load in order to allow for a factor of safety.

When examining the strain energy density distribution for the optimum design it can be seen that the densities are not exactly equal for all the elements. This may be due to the inability of the algorithm to calculate the nonlinear critical load and the associated displacement pattern accurately. It was found that if the nonlinear critical point was a simple limit point then the present algorithm gave an optimum design with a fairly uniform strain energy density distribution. Some structures, like the shallow dome under a distributed load, exhibit both a bifurcation point and a limit point type of instability at load levels that are not too far apart. For such structures the present algorithm gave a near optimum design. By insisting on a lower weight in successive iterations, it was not possible to obtain a uniform strain energy density distribution. The reason for this tendency appears to stem from the fact that the structure was designed on the basis of a single dominant displacement mode. In order to satisfy the optimality criterion in all the members, it will be necessary to use all possible active displacement modes at the nonlinear critical point and treat the problem as a multiple constrained problem. The slight nonuniformity of the strain energy density distribution for the near optimum design of the shallow truss of example 3 may in part be due to such behavior.

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